

A New Monotone Quantity along the Inverse Mean Curvature Flow in \mathbb{R}^n

Kwok-Kun Kwong* Pengzi Miao†

Abstract

We find a new monotone increasing quantity along smooth solutions to the inverse mean curvature flow in \mathbb{R}^n . As an application, we derive a sharp geometric inequality for mean convex, star-shaped hypersurfaces which relates the volume enclosed by a hypersurface to a weighted total mean curvature of the hypersurface.

1 Statement of the Result

Monotone quantities along hypersurfaces evolving under the inverse mean flow have many applications in geometry and relativity. In [3], Huisken and Ilmanen applied the monotone increasing property of Hawking mass to give a proof of the Riemannian Penrose Inequality. In a recent paper [1], Brendle, Hung and Wang discovered a monotone decreasing quantity along the inverse mean curvature flow in Anti-Desitter-Schwarzschild manifolds and used it to establish a Minkowski-type inequality for star-shaped hypersurfaces.

In this note, we provide a new monotone increasing quantity along smooth solutions to the inverse mean curvature flow in \mathbb{R}^n :

Theorem 1. *Let Σ be a smooth, closed, embedded hypersurface with positive mean curvature in \mathbb{R}^n . Let I be an open interval and $X : \Sigma \times I \rightarrow \mathbb{R}^n$ be a smooth map satisfying*

$$\frac{\partial X}{\partial t} = \frac{1}{H}\nu, \quad (1.1)$$

*School of Mathematical Sciences, Monash University, Victoria 3800, Australia. E-mail: kwok-kun.kwong@monash.edu

†Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA. E-mail: pengzim@math.miami.edu

where H is the mean curvature of the surface $\Sigma_t = X(\Sigma, t)$ and ν is the outward unit normal vector to Σ_t . Let Ω_t be the bounded region enclosed by Σ_t and $r = r(x)$ be the distance from x to a fixed point O . Then the function

$$Q(t) = e^{-\frac{n-2}{n-1}t} \left[n \text{Vol}(\Omega_t) - \frac{1}{n-1} \int_{\Sigma_t} r^2 H d\mu \right] \quad (1.2)$$

is monotone increasing and $Q(t)$ is a constant function if and only if Σ_t is a round sphere for each t . Here $\text{Vol}(\Omega)$ denotes the volume of a bounded region Ω and $d\mu$ denotes the volume form on a hypersurface.

As an application, we derive a sharp inequality for star-shaped hypersurfaces in \mathbb{R}^n which relates the volume enclosed by a hypersurface to an r^2 -weighted total mean curvature of the hypersurface.

Theorem 2. *Let Σ be a smooth, star-shaped, closed hypersurface embedded in \mathbb{R}^n with positive mean curvature. Then*

$$n \text{Vol}(\Omega) \leq \frac{1}{n-1} \int_{\Sigma} r^2 H d\mu \quad (1.3)$$

where $\text{Vol}(\Omega)$ is the volume of the region Ω enclosed by Σ , r is the distance to a fixed point O and H is the mean curvature of Σ . Furthermore, equality in (1.3) holds if and only if Σ is a sphere centered at O .

We give some remarks about Theorem 1 and Theorem 2. The discovery of the monotonicity of $Q(t)$ in Theorem 1 is motivated by the recent work of Brendle, Hung and Wang in [1, Section 5]. To prove Theorem 1, we also need a result due to Ros [7] which was proved using Reilly's formula [6]. Having known $Q(t)$ is monotone increasing, to prove Theorem 2, it may be attempting to ask whether $\lim_{t \rightarrow \infty} Q(t) = 0$. We do not know if this is true because both $\text{Vol}(\Omega_t)$ and $\int_{\Sigma_t} r^2 H d\mu$ grow like $e^{\frac{n}{n-1}t}$ when $\{\Sigma_t\}$ are spheres while there is only a factor of $e^{-\frac{n-2}{n-1}t}$ in (1.2). Instead, we take an alternate approach by first proving Theorem 2 for a convex hypersurface Σ . The proof in that case again makes use of Reilly's formula. When Σ is merely assumed to be mean convex and star-shaped, we prove Theorem 2 by reducing it to the convex case using solutions to the inverse mean curvature flow provided by the works of Gerhardt [2] and Urbas [8]. If a stronger result of Huisken and Ilmanen in [4] is applied, Theorem 2 indeed can be shown to hold for star-shaped surfaces with nonnegative mean curvature. We will discuss this case in the end.

2 Proof of the Theorems

Given a compact Riemannian manifold (Ω, g) with boundary Σ , we recall that Reilly's formula [6] asserts

$$\begin{aligned} & \int_{\Omega} |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u \rangle + \text{Ric}(\nabla u, \nabla u) dV \\ &= \int_{\Sigma} (\Delta u) \frac{\partial u}{\partial \nu} - \text{III}(\nabla^{\Sigma} u, \nabla^{\Sigma} u) - 2(\Delta_{\Sigma} u) \frac{\partial u}{\partial \nu} - H \left(\frac{\partial u}{\partial \nu} \right)^2 d\mu. \end{aligned} \quad (2.1)$$

Here u is a smooth function on Ω ; ∇^2 , Δ and ∇ denote the Hessian, the Laplacian and the gradient on Ω ; Δ_{Σ} and ∇^{Σ} denote the Laplacian and the gradient on Σ ; ν is the unit outward normal vector to Σ ; III and H are the second fundamental form and the mean curvature of Σ with respect to ν ; and Ric is the Ricci curvature of g .

To prove Theorem 1, we need a result of Ros [7], which was proved by choosing $\Delta u = 1$ on Ω and $u = 0$ at Σ in the above Reilly's formula.

Theorem 3 (Ros [7]). *Let (Ω, g) be an n -dimensional compact Riemannian manifold with nonnegative Ricci curvature with boundary Σ . Suppose Σ has positive mean curvature H , then*

$$n \text{Vol}(\Omega) \leq (n-1) \int_{\Sigma} \frac{1}{H} d\mu \quad (2.2)$$

and equality holds if and only if (Ω, g) is isometric to a round ball in \mathbb{R}^n .

Proof of Theorem 1. We use $'$ to denote differentiation w.r.t t . Some basic formulas along the inverse mean curvature flow (1.1) in \mathbb{R}^n are

$$H' = -\Delta_{\Sigma_t} \left(\frac{1}{H} \right) - \frac{|\text{III}|^2}{H}, \quad d\mu' = d\mu, \quad \text{Vol}(\Omega_t)' = \int_{\Sigma_t} \frac{1}{H} d\mu. \quad (2.3)$$

Let $u = r^2$, then u satisfies

$$\nabla^2 u = 2g \quad \text{and} \quad \Delta u = 2n, \quad (2.4)$$

where g is the Euclidean metric. Now

$$\left(\int_{\Sigma_t} u H d\mu \right)' = \int_{\Sigma_t} (u' H + u H' + u H) d\mu. \quad (2.5)$$

Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product. By (2.3), (2.4) and the divergence theorem, we have

$$\int_{\Sigma_t} u' H d\mu = \int_{\Sigma_t} \langle \nabla u, \frac{1}{H} \nu \rangle H d\mu = \int_{\Omega_t} \Delta u dV = 2n \text{Vol}(\Omega_t). \quad (2.6)$$

By (2.4), we also have

$$\Delta_{\Sigma_t} u = \Delta u - H \frac{\partial u}{\partial \nu} - \nabla^2 u(\nu, \nu) = 2(n-1) - H \frac{\partial u}{\partial \nu},$$

which together with (2.3) - (2.4) implies

$$\begin{aligned} \int_{\Sigma_t} u H' d\mu &= \int_{\Sigma_t} \left(-\frac{\Delta_{\Sigma_t} u}{H} - \frac{u |\mathbb{I}\mathbb{I}\mathbb{I}|^2}{H} \right) d\mu \\ &= \int_{\Sigma_t} \left(-\frac{2(n-1)}{H} + \frac{\partial u}{\partial \nu} - \frac{u |\mathbb{I}\mathbb{I}\mathbb{I}|^2}{H} \right) d\mu \\ &= - \int_{\Sigma_t} \frac{2(n-1)}{H} d\mu + 2n \text{Vol}(\Omega_t) - \int_{\Sigma_t} \frac{u |\mathbb{I}\mathbb{I}\mathbb{I}|^2}{H} d\mu. \end{aligned} \quad (2.7)$$

Substituting (2.6) and (2.7) into (2.5) yields

$$\begin{aligned} \left(\int_{\Sigma_t} u H d\mu \right)' &= 4n \text{Vol}(\Omega_t) + \int_{\Sigma_t} \left[-\frac{2(n-1)}{H} - \frac{u |\mathbb{I}\mathbb{I}\mathbb{I}|^2}{H} + u H \right] d\mu \\ &\leq 4n \text{Vol}(\Omega_t) + \int_{\Sigma_t} \left[-\frac{2(n-1)}{H} - \frac{u H}{n-1} + u H \right] d\mu \\ &= 4n \text{Vol}(\Omega_t) + \int_{\Sigma_t} \left[-\frac{2(n-1)}{H} + \frac{n-2}{n-1} u H \right] d\mu \\ &\leq 4n \text{Vol}(\Omega_t) - 2n \text{Vol}(\Omega_t) + \frac{n-2}{n-1} \int_{\Sigma_t} u H d\mu \\ &= 2n \text{Vol}(\Omega_t) + \frac{n-2}{n-1} \int_{\Sigma_t} u H d\mu \end{aligned} \quad (2.8)$$

where we have used $|\mathbb{I}\mathbb{I}\mathbb{I}|^2 \geq \frac{1}{n-1} H^2$ in line 2 and Theorem 3 in line 4. On the other hand, by Theorem 3 again, we have

$$\text{Vol}(\Omega_t)' = \int_{\Sigma_t} \frac{1}{H} d\mu \geq \frac{n}{n-1} \text{Vol}(\Omega_t). \quad (2.9)$$

It follows from (2.8) and (2.9) that

$$\left[n(n-1)\text{Vol}(\Omega_t) - \int_{\Sigma_t} uH d\mu \right]' \geq \frac{n-2}{n-1} \left[n(n-1)\text{Vol}(\Omega_t) - \int_{\Sigma_t} uH d\mu \right]$$

or equivalently

$$\left[e^{-\frac{n-2}{n-1}t} \left(n\text{Vol}(\Omega_t) - \frac{1}{n-1} \int_{\Sigma_t} r^2 H d\mu \right) \right]' \geq 0. \quad (2.10)$$

We conclude that $Q(t)$ is monotone increasing, moreover $Q(t)$ is a constant function if and only if equalities in (2.8) and (2.9) hold. By Theorem 3, we know these equalities hold if and only if Σ_t is a round sphere for all t . This completes the proof of Theorem 1. \square

Next, we prove Theorem 2 in the case that Σ is a convex hypersurface.

Proposition 1. *Let Σ be a smooth, closed, convex hypersurface embedded in \mathbb{R}^n . Then*

$$n\text{Vol}(\Omega) \leq \frac{1}{n-1} \int_{\Sigma} r^2 H d\mu \quad (2.11)$$

where $\text{Vol}(\Omega)$ is the volume of the region Ω enclosed by Σ , r is the distance to a fixed point O and H is the mean curvature of Σ . Moreover, equality in (2.11) holds if and only if Σ is a sphere centered at O .

Remark 4. Proposition 1 generalizes an inequality of the first author in [5, Theorem 3.2 (1)].

Proof. Apply Reilly's formula (2.1) to the Euclidean region Ω and choose $u = r^2$, we have

$$4n(n-1)\text{Vol}(\Omega) = \int_{\Sigma} \text{III}(\nabla^{\Sigma} u, \nabla^{\Sigma} u) + 2(\Delta_{\Sigma} u) \frac{\partial u}{\partial \nu} + H \left(\frac{\partial u}{\partial \nu} \right)^2 d\mu$$

where

$$\Delta_{\Sigma} u = \Delta u - H \frac{\partial u}{\partial \nu} - \nabla^2 u(\nu, \nu) = 2(n-1) - H \frac{\partial u}{\partial \nu}.$$

Therefore,

$$\int_{\Sigma} H \left(\frac{\partial u}{\partial \nu} \right)^2 d\mu = \int_{\Sigma} \text{III}(\nabla^{\Sigma} u, \nabla^{\Sigma} u) d\mu + 4n(n-1)\text{Vol}(\Omega). \quad (2.12)$$

Since Σ is convex, $\mathbb{I}(\cdot, \cdot)$ is positive definite. Hence, (2.12) implies

$$n(n-1)\text{Vol}(\Omega) \leq \frac{1}{4} \int_{\Sigma} H \langle \nabla(r^2), \nu \rangle^2 d\mu \leq \int_{\Sigma} H r^2 d\mu. \quad (2.13)$$

When $n(n-1)\text{Vol}(\Omega) = \int_{\Sigma} H r^2 d\mu$, we must have $\mathbb{I}(\nabla^{\Sigma} u, \nabla^{\Sigma} u) = 0$, hence $\nabla^{\Sigma} u = 0$. This implies that $u = r^2$ is a constant on Σ , which shows that Σ is a sphere centered at O . \square

To deform a star-shaped hypersurface to a convex hypersurface through the inverse mean curvature flow, we make use of a special case of a general result of Gerhard [2] and Urbas [8].

Theorem 5 (Gerhardt [2] and Urbas [8]). *Let Σ be a smooth, closed hypersurface in \mathbb{R}^n with positive mean curvature, given by a smooth embedding $X_0 : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$. Suppose Σ is star-shaped with respect to a point P . Then the initial value problem*

$$\begin{cases} \frac{\partial X}{\partial t} = \frac{1}{H} \nu \\ X(\cdot, 0) = X_0(\cdot) \end{cases} \quad (2.14)$$

has a unique smooth solution $X : \mathbb{S}^{n-1} \times [0, \infty) \rightarrow \mathbb{R}^n$, where ν is the unit outer normal vector to $\Sigma_t = X(\mathbb{S}^{n-1}, t)$ and H is the mean curvature of Σ_t . Moreover, Σ_t is star-shaped with respect to P and the rescaled hypersurface $\tilde{\Sigma}_t$, parametrized by $\tilde{X}(\cdot, t) = e^{-\frac{t}{n-1}} X(\cdot, t)$, converges to a sphere centered at P in the C^∞ topology as $t \rightarrow \infty$.

Now we can complete the proof of Theorem 2.

Proof of Theorem 2. By Theorem 5, there exists a smooth solution $\{\Sigma_t\}$ to the inverse mean curvature flow with initial condition Σ . Moreover, the rescaled hypersurface $\tilde{\Sigma}_t = \{e^{-\frac{t}{n-1}} x \mid x \in \Sigma_t\}$ converges exponentially fast in the C^∞ topology to a sphere. In particular, $\tilde{\Sigma}_t$ and hence Σ_t , must be convex for large t .

Let T be a time when Σ_T becomes convex. By Proposition 1, we have

$$n\text{Vol}(\Omega_T) \leq \frac{1}{n-1} \int_{\Sigma_T} r^2 H d\mu,$$

i.e. $Q(T) \leq 0$. By Theorem 1, we know $Q(t)$ is monotone increasing, hence

$$Q(0) \leq Q(T) \leq 0$$

which proves (1.3).

If the equality in (1.3) holds, then $Q(0) = 0$. It follows from the monotonicity of $Q(t)$ and the fact $Q(t) \leq 0$ for large t that

$$Q(t) = 0, \forall t.$$

By Theorem 1, this implies that Σ_t is a sphere for each t . By Proposition 1, Σ_t is a sphere centered at O for large t . Therefore, we conclude that the initial hypersurface Σ is a sphere centered at O . □

3 The case of nonnegative mean curvature

Suppose Σ is a star-shaped hypersurface with nonnegative mean curvature in \mathbb{R}^n . By approximating Σ with star-shaped hypersurfaces with positive mean curvature, it is not hard to see that the inequality (1.3) still holds for Σ . (For instance, such an approximation can be provided by the short time solution to the mean curvature flow with initial condition Σ .)

To see that the rigidity part of (1.3) also holds for such a Σ , we resort to a result of Huisken and Ilmanen in [4, Theorem 2.5]:

Theorem 6 (Huisken and Ilmanen [4]). *Let $X_0 : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$ be an embedding such that $\Sigma = X_0(\mathbb{S}^{n-1})$ is a C^1 , star-shaped hypersurface with measurable, bounded, nonnegative weak mean curvature. Then*

$$\frac{\partial X}{\partial t} = \frac{1}{H}\nu \tag{3.1}$$

has a smooth solution $X : \mathbb{S}^{n-1} \times (0, \infty) \rightarrow \mathbb{R}^n$ such that as $t \rightarrow 0+$, the hypersurface $\Sigma_t = X(\mathbb{S}^{n-1}, t)$ converges to Σ uniformly in C^0 .

Remark 7. In the above theorem, if the initial surface Σ is assumed to be smooth, the same proof in [4] together with the upper estimate of H for smooth solutions (c.f. [3, (1.4)]) shows that as $t \rightarrow 0+$, Σ_t converges to Σ in $W^{2,p}$ norm for any $1 < p < \infty$. On the other hand, by Theorem 5, Σ_t converges to a sphere in the C^∞ topology after rescaling, as $t \rightarrow \infty$. In particular, Σ_t is convex for large enough $t > 0$.

It follows from Theorem 1, Proposition 1, Theorem 6 and Remark 7 that

Theorem 8. *Let Σ be a smooth, star-shaped, closed hypersurface embedded in \mathbb{R}^n with nonnegative mean curvature. Then*

$$n\text{Vol}(\Omega) \leq \frac{1}{n-1} \int_{\Sigma} r^2 H d\mu \quad (3.2)$$

where $\text{Vol}(\Omega)$ is the volume of the region Ω enclosed by Σ , r is the distance to a fixed point O and H is the mean curvature of Σ . Furthermore, equality in (1.3) holds if and only if Σ is a sphere centered at O .

References

- [1] S. Brendle, P.-K. Hung, and M.-T. Wang, *A Minkowski-type inequality for hypersurfaces in the Anti-Desitter-Schwarzschild manifold*, arXiv:1209.0669.
- [2] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299–314.
- [3] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.
- [4] G. Huisken and T. Ilmanen, *Higher regularity of the inverse mean curvature flow*, J. Differential Geom. **80** (2008), no. 3, 433–451.
- [5] K.-K. Kwong, *On convex hypersurfaces in space forms and eigenvalues estimates for differential forms*, arXiv:1207.3999.
- [6] R.C. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. **26** (1977), 459–472.
- [7] A. Ros, *Compact Hypersurfaces with Constant Higher Order Mean Curvatures*, Rev. Mat. Iberoamericana **3** (1987), no. 3, 447–453.
- [8] J.I.E. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 3, 355–372.